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NOTES ON BLAND'S PIVOTING RULE*

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Dedicated to the memory of D. Ray Fulkerson

Recently R.G. Bland proposed two new rules for pivot selection in the simplex method. These elegant rules arise from Bland's work on oriented matroids; their virtue is that they never lead to cycling. We investigate the efficiency of the first of them. On randomly generated problems with 50 nonnegative variables and 50 additional inequalities, Bland's rule requires about 400 iterations on the average; the corresponding figure for the popular "largest coefficient" rule is only about 100. Comparable behaviour seems to persist even on highly degenerate problems. On the theoretical side, we analyse the performance of Bland's rule on the classical Klee–Minty examples: for problems with n nonnegative variables and n additional inequalities, the number of iterations is bounded from below by the n-th Fibonacci number.

Key words: Simplex Method, Pivoting, Number of Iterations, Degeneracy and Cycling, Monte Carlo Experiments, Klee–Minty Examples.

1. Introduction

The commonly used pivoting rules for the simplex method can lead to cycling: examples of that phenomenon have been constructed by Hoffman [5], Beale [1] and others. Although cycling is virtually unknown in practice, its potential threat is unpleasant from the theoretical point of view. Fortunately, it can be avoided by precautions such as the perturbation technique [3] or, equivalently, the lexicographic technique [4]. Quite recently, R.G. Bland [2] dealt cycling an impressive *coup de grâce* by presenting an elegant proof that a certain simple and natural pivoting rule never leads to cycling. Bland's rule is extremely simple indeed:

(i) among all the candidates to enter the basis, select the variable with the smallest subscript,

(ii) if two or more variables compete for leaving the basis, select again the variable with the smallest subscript.

The importance of this "smallest subscript" rule is primarily theoretical: it was Bland's work on oriented matroids that led him to the discovery of the rule. (In

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fact, [2] contains *two* different pivoting rules that never lead to cycling: the second of them is slightly more complicated and we shall not discuss it here.) It seems likely that Bland's pivoting rules may find applications in other theoretical studies involving the simplex method.

Even though the smallest subscript rule was not meant to compete with other pivoting rules in actual computations, it makes sense to ask how its performance compares with those of the commonly used pivoting rules. (On the intuitive level, it may seem that the odds are stacked against the smallest subscript rule since that rule ignores much of the valuable information contained in a simplex tableau: taking into account only the *signs* of the coefficients in the objective row, it disregards their *magnitudes*. In our opinion, such feeling does not carry much weight unless confirmed experimentally: one might as well argue that the smallest subscript rule should have the edge because of its highly systematic nature. After all, doesn't it have the edge where cycling is concerned?) A satisfactory approximation to the precise answer can be provided by Monte Carlo experiments. In the early Sixties, Kuhn and Quandt [8] carried out such experiments to compare the performances of nine different pivoting rules. Following their example, we have compared the smallest subscript rule with two of the most popular pivoting rules:

(α) among all the candidates to enter the basis, select that variable whose "relative cost" has the largest absolute value,

(β) among all the candidates to enter the basis, select that variable whose entrance will bring about the largest increase of the objective function. The results are reported in the next section: the smallest subscript rule (γ) did not fare too well. For problems with fifty nonnegative variables and fifty additional inequality constraints, it required close to four hundred iterations on the average; the average number of iterations required by (α) to solve the same problems was less than one hundred. Experimenting with some highly degenerate problems, we were surprised to find that (α) was still better than (γ). In fact, the ratios of the corresponding numbers of iterations were quite close to those for nondegenerate problems of comparable sizes.

Another question of possible interest is theoretical: could it be that the number of iterations required by (γ) is bounded from above by a fixed polynomial in the size of the problem? For the pivoting rule (α), the same question was answered negatively in the pioneering paper [7] by Klee and Minty; later on, a similar result for (β) was established by Jeroslow [6]. Klee and Minty begin their exposition by presenting very simple problems with n nonnegative variables and n additional inequalities for which the simplex method may take $2^n - 1$ iterations if the entering variables are chosen in a consistently ill-advised way. We shall prove that the number of iterations required by (γ) to solve these problems is bounded from below by the n-th Fibonacci number. The same holds for the completely degenerate problems resulting from the original examples when each right-hand side is replaced by a zero.